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Prime Alternative Rings, III

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1. INTRODUCTION

Throughout this paper 'ring' means 'alternative ring,' and R denotes a ring. We write $A \leqslant R$ to mean that A is an ideal of R. $N(R)$, $Z(R)$, $D(R)$, $U(R)$, the nucleus, center, associator ideal, and maximum nuclear ideal of R, respectively, are defined as in [9, 2.1].

1.1. In [3] Herstein produces the elegant method of 'reasonable elements' to show that if R is simple but neither associative nor nil, and if char $R \neq 2$, then R is a Cayley–Dickson algebra (CD algebra) over F. In this paper we use Herstein's method to extend the results on prime rings which we obtained in [9]. There we showed that a prime but not associative ring R with $3R \neq (0)$ is a CD ring (see [9, 3.3] for the definition); here we show that restrictions on characteristic can be replaced by restrictions of other kinds. However, it is still not known whether *all* prime rings are CD rings or associative.

1.2. In Section 2 we use Herstein's method to prove that an algebra R over F is quadratic over F, if R satisfies certain conditions which later turn out to be fairly weak. In Section 3 we use these results to prove that results analogous to those of [9] hold for rings satisfying various conditions of 'weak primeness' type [9, Section 2], if the restrictions on characteristic given in [9] are replaced in each case by some restriction concerning the existence of reasonable elements. In Section 4 we show how these (rather artificial) conditions can be avoided if we strengthen one of our other hypotheses, that R be semiprime, to the hypothesis that R be free of nil ideals. This follows from known results (4.2, 4.3, 4.4) whose proofs, once discovered, are not difficult. Using a couple of much harder known results (4.8, and Theorems A and C of [9]), we then show that 'free of nil ideals' can be replaced by 'free of locally nilpotent ideals'. For instance, if R has no nonzero locally nilpotent ideals, and $Z(R)$ contains no R-zero-divisors, then R is a CD ring or associative.

2. Quadratic Algebras

2.1. For the following result we assume (as nearly as possible) the minimum under which Herstein's argument for [3, Theorem 7], will work. Although our assumptions are rather artificial, this result is pivotal for all that follows.

Following Herstein, we say $a \in R$ is *reasonable* provided for some $b \in R$ we have $(a, b)^4 \neq 0$, where we write (a, b) for $ab - ba$. In this situation we will also say that the pair $\langle a, b\rangle$ is a *reasonable pair*.

Proposition. 2.2 *Let S be an alternative algebra with* 1 *over a field F, where* $\operatorname{card} F > 5$. *Suppose that S has some reasonable pair $\langle a, b\rangle$, and that for all reasonable pairs $\langle x, y\rangle$ we have $(x, y)^2 \in F$. Then S is quadratic over F.*

We will consistently identify the subring $F \cdot 1$ of S with F. The conclusion means that, given $x \in S$, there exist $\theta, \gamma \in F$ such that $x^2 - \theta x + \gamma = 0$. Our proof will differ slightly from Herstein's, in order to bypass his Lemma 3.8. We give it in stages.

1. *If $y \in S$, $c \in S$, and $c^4 \neq 0$, then we may find distinct $\beta_i \in F$ $(i = 1, 2)$ such that $(y + \beta_i c)^4 \neq 0$.*

For let us define a function $h : F \to S$ by

$$h(\beta) = (y + \beta c)^4 = b_0 + b_1\beta + \cdots + b_4\beta^4.$$

If $h(\beta_i) = 0$ for distinct $\beta_i : i = 0, 1,..., 4$, we may write $\mathbf{bM} = 0$, where $\mathbf{b} = < b_0 ,..., b_4 > \in S^4$, and $\mathbf{M} = (\beta_i{}^j) \in F_5$. Now the matrix $\mathbf{M}$ has a vanderMonde determinant, so is nonsingular. Hence $\mathbf{0} = (\mathbf{bM})\,\mathbf{M}^{-1} = \mathbf{b}$, since $b_i 1 = b_i$. But this contradicts $b_4 = c^4 \neq 0$. Thus $h(\beta) = 0$ for at most four values of β, leaving at least two β's for which $h(\beta) \neq 0$.

2. *If $\langle a, b\rangle$ is a reasonable pair, and $x \in S$ is arbitrary, we may find distinct $\beta_i \in F$ $(i = 1, 2)$ such that $\langle \beta_i a + x, b\rangle$ is a reasonable pair.*

This follows from 1 on taking $c = (a, b)$; $y = (x, b)$.

3. *If $x \in S$ is arbitrary, and b is reasonable, then $(x, b)^2 \in F$.*

Choose $a \in S$ such that $\langle a, b\rangle$ is a reasonable pair. By 2, we may choose distinct $\beta_i \in F$ such that $\langle \beta_i a + x, b\rangle$ are reasonable pairs. Thus $(\beta_i a + x, b)^2 = \alpha_i \in F$, or

$$t^2 + \beta_i z + \beta_i{}^2\gamma = \alpha_i ,$$

where $t = (x, b)$; $z = t(a, b) + (a, b)t$; $(a, b)^2 = \gamma \in F$. Eliminating z, we find $(\beta_2 - \beta_1) t^2 \in F$, whence $t^2 \in F$.

4. *If be is reasonable, then b is quadratic over F.*

Choose $a \in S$ so that $\langle a, b \rangle$ is a reasonable pair, and set $(a, b) = t$. By hypothesis $t^2 = \theta \in F$, with $\theta \neq 0$. Also $tb = (ab, b)$, so that, by 3, $(tb)^2 = \gamma' \in F$. Similarly, $t + tb = (a + ab, b)$, so that $(t + tb)^2 \in F$. On expanding this, we find

$$t^2 + t^2 b + tbt + (tb)^2 \in F,$$

or

$$\alpha' + \theta b + tbt = 0 \quad (\text{some } \alpha' \in F).$$

So

$$\theta b^2 + \alpha' b + \gamma' = 0; \quad \theta \neq 0,$$

whence

$$b^2 + \alpha b + \gamma = 0 \; (\alpha = \theta^{-1}\alpha', \text{ etc.}).$$

5. *Every $x \in S$ is quadratic over F.*

Let $\langle a, b \rangle$ be any reasonable pair. By $a \leftrightarrow b$ symmetry, we can assume that either $(x, a) \neq 0$ or $(x, b) = 0$. Now choose β_i as in 2. Then by 4

$$(x + \beta_i a)^2 + \alpha_i (x + \beta_i a) \in F \qquad (i = 1, 2)$$

for suitable $\alpha_i \in F$. Since also $a^2 \in F + Fa$, this yields

$$x^2 + \alpha_i x + \beta_i y \in F + Fa,$$

where $y = xa + ax$. On eliminating y and dividing through by $(\beta_2 - \beta_1)$, we find

$$x^2 + \alpha x + \gamma = \theta a,$$

for suitable $\alpha, \gamma, \theta \in F$. If $(x, b) = 0$, we commute this relation with b to find $\theta(a, b) = 0$. If $(x, a) \neq 0$ we commute it with x to find $\theta(a, x) = 0$. So in any case $\theta = 0$, and $x^2 + \alpha x + \gamma = 0$.

Note 2.3. This result may be sharpened as follows: Suppose F is merely a subring of the center $Z(S)$ of the ring S, and $\alpha r \neq 0$ for $0 \neq \alpha \in F$; $0 \neq r \in S$. If the other assumptions remain, then S is quadratic over F in the sense that, for all $x \in S$, we have a relation of the form

$$\alpha x^2 + \beta x + \gamma = 0; \qquad \alpha, \beta, \gamma \in F; \qquad \alpha \neq 0.$$

2.4. It can be shown by more complicated arguments that the conclusion of Proposition 2.2 still holds when the restriction $\operatorname{card} F > 5$ is relaxed to $\operatorname{card} F > 2$. The result holds even if $F = GF2$, but a non-elementary proof is then required.

For the purposes of this paper the 'non-linear' hypothesis concerning reasonable elements which we made in Proposition 2.2 (viz., $(x, y)^4 \neq 0$ implies $(x, y)^2 \in F$) is not of independent interest, and we now give a result using the stronger but more natural hypothesis that $N(R) = Z(R) = F \cdot 1$. Since this hypothesis is linear (i.e., preserved under scalar extension), we need no restriction on $\operatorname{card} F$. For later use we give a more detailed conclusion than that of Proposition 2.2.

PROPOSITION 2.5. *Suppose R is an algebra over a field F such that $N(R) = Z(R) = F \cdot 1 = F$, say. Suppose also R has a reasonable element. Then there exist functions $t, n : R \to F$ such that*

(1) *for all $x \in R$,* $\quad x^2 - t(x)x + n(x) = 0,$

(2) *t is linear over F,*

(3) *n is a composing quadratic form.*

(4) *If $W = \{w \in R : n(w + x) = n(x)$ for all $x \in R\}$,*
then W is the maximum nil ideal of R.

Proof. Let K be any extension field of F such that $\operatorname{card} K > 5$. Set $S = R \otimes_F K$. It is easily verified that S is alternative, and that $N(S) = N(R) \otimes_F K = K = Z(S)$. Also if $\langle a, b \rangle$ is a reasonable pair in R, then $\langle a \otimes 1, b \otimes 1 \rangle$ is a reasonable pair in S. Next, S satisfies the identity

$$(v^4, r, s) = v(v^2, r, s) = 0$$

for all $r, s \in S$, and all $v = (x, y) \in S$ [3, p. 380]. Thus, $v^4 \in N(S) = K$, and if $v^4 \neq 0$ this shows that v is a nonzero-divisor. Hence $(v^2, r, s) = 0$; so $v^2 \in N(S) = K$. That is, $(x, y)^4 \neq 0$ implies $(x, y)^2 \in K$. So S satisfies all the conditions of Proposition 2.2, and so is quadratic over K.

If $x \in S$, $x \notin K$, it is immediate that x satisfies a *unique* relation $x^2 - \theta x + \gamma = 0$ with $\theta, \gamma \in K$. In this case we define $t_K(x) = \theta$; $n_K(x) = \gamma$. If $x \in K$ we define $t_K(x) = 2x$; $n_K(x) = x^2$. We have thus defined functions $t_K, n_K : S \to K$.

Since $\operatorname{card} K > 2$, a linearization argument shows that t_K is linear over K. It follows easily that if $g_K(a, b) = n_K(a + b) - n_K(a) - n_K(b)$, then g_K is bilinear over K. Indeed, if we define $* : S \to S$ by $x^* = t_K(x) - x$, then $*$ is K-linear, and $g(a, b) = a^*b + b^*a$. Together with the obvious $n_K(\alpha x) = \alpha^2 n_K(x)$, this shows that n_K is a quadratic form. A further argument shows that n_K composes, i.e., $n_K(ab) = n_K(a) \cdot n_K(b)$.

(For all this see, for example, [3, pp. 391–393]. On the last line of p. 392 and third line of p. 393, $t(a)\,ab$ is a misprint for $t(a)\,bab$. The argument there is stated for associative rings, but goes through for arbitrary rings in view of Artin's theorem.)

Next, set $W_K = \{w \in S : n_K(w + x) = n_K(x) \text{ for all } x \in S\}$. The argument used in [3, p. 390], goes through to show that $W_K \leqslant S$, since card $K > 2$. If $w \in W_K$, then $n_K(w) = 0$; so $w^2 = \theta w$ with $\theta \in K$. Then, $(w + 1)^2 = (\theta + 2)(w + 1) - (\theta + 1)$, and $t_K(w + 1) = t_K(w) + t_K(1) = \theta + 2$ yields $n_K(w + 1) = \theta + 1$. But by definition of W_K, we have $n_K(w + 1) = n_K(1) = 1$. Thus, $\theta = 0$, and W_K is nil.

On setting $t = t_K \restriction R$; $n = n_K \restriction R$, we have 1, 2, 3 of the conclusions, and in view of $W = W_K \cap R$ we have most of 4. It remains only to show that W is the *largest* nil ideal in R. Thus, let Y be any nil ideal of R, and let $y \in Y$ be given. Clearly $y^2 = 0$. Then $0 = n(0) = n(y^2) = [n(y)]^2$ yields $n(y) = 0$, whence also $t(y) = 0$, and $y^* = -y$. Now given any $x \in R$ we have $g(x, y) = x^*y + y^*x = x^*y - yx \in F \cap Y = (0)$. So

$$0 = g(x, y) - n(x) - n(y) = n(x + y) - n(x), \qquad \text{and} \quad y \in W.$$

So $Y \subseteq W$, as required.

2.6. The function n above is said to be *nonsingular* provided $n(x) = 0$ and $g(x, R) = (0)$ implies $x = 0$. It is immediate that n is nonsingular if and only if $W = (0)$. Furthermore, it is clear that for any $x \in R$ and $w \in W$ we have

$$(x + w)^2 - t(x)(x + w) + n(x) = 0.$$

Thus, t and n induce in a natural way functions $\bar{t}$, $\bar{n}$ on the difference algebra $\bar{R} = R - W$, and $\bar{R}$ is quadratic over F. Clearly also, $W(\bar{R}) = (0)$. So $W = W(R)$ is the *n-radical* of R in the sense that the induced function on $R - W$ is nonsingular, and W is the smallest ideal of R having this property.

It is quite possible to have $W \neq (0)$ in the situation of Proposition 2.5: consider the example in [9, 4.2]. However, $W = (0)$ if R is semiprime, in view of our next result, which we shall need in Section 3. Recall that an ideal T of R is *trivial* provided $T \neq (0) = T^2$, and R is *semiprime* provided it has no trivial ideals.

LEMMA 2.7. *Suppose the algebra R over F has an ideal $W \neq (0)$ which is nil of index 2. Then W contains a trivial ideal of R.*

Proof. If $p, q, r, s \in W$, then $pq + qp = 0$, whence $p \cdot qr + q \cdot pr = (pq + qp)r = 0 = pq \cdot r + pr \cdot q$ similarly. Hence

$$pq \cdot r = -pr \cdot q = q \cdot pr = -p \cdot qr.$$

But now $pq \cdot rs = -p \cdot q(rs) = p \cdot (qr)s = -p(qr) \cdot s = (pq)r \cdot s = -pq \cdot rs$. So $2W^2W^2 = (0)$, whence $[(2W)^2]^2 = (0)$. We are thus done unless $2W = (0)$. So suppose from now on that $2W = (0)$. Then $pq + qp = 0$ shows that W is commutative, and $pq \cdot r = -p \cdot qr$ shows that it is associative. So $W = Z(W)$.

Now if the conclusion of the lemma is false, then R is W-semiprime, so that $W = Z(W) = W \cap Z(R)$ by [9, 3.2b]. Since $W \neq (0)$, we can find $0 \neq t \in W \cap Z(R)$, and then $t^2 = 0$ implies that $T = Ft + Rt$ is a trivial ideal of R contained in W. This gives us the required contradiction.

Lemma 2.7 has the following important

COROLLARY 2.8. *Suppose R satisfies the hypotheses of Proposition* 2.5, *and also is semiprime. Then R is a* CD *algebra over F.*

Proof. By Lemma 2.7 $W = (0)$; so n is a nonsingular composing quadratic form on R. But now an easy argument due to Kaplansky [4, p. 959] shows that R is simple or $R \simeq F \otimes F$. But R is not associative, or $R = N(R) = Z(R)$ contradicts the existence of a reasonable element. So R is a simple alternative but not associative ring with 1; thus a CD algebra over $Z(R) = F$ [9, 3.2h].

Digression 2.9. Lemma 2.7 (for rings) is the extension to arbitrary rings of a special case (viz., $n = 2$) of a theorem of Levitzki for associative rings [3, p. 385]: if A is a nil right ideal of bounded index, then A contains a nilpotent right ideal of R. It would be interesting to know whether this result in full generality holds for arbitrary R. Although it is of no relevance to the rest of this paper, we now give a partial result in this direction.

THEOREM L. *Let A be a right ideal of the ring R. If A is nil of bounded index, then either $3A = (0)$, or A contains a trivial right ideal of R.*

(I do not know whether the possibility $3A = (0)$ can be dispensed with.) The proof splits naturally into two cases.

Proof. 1. *R is also A-purely alternative.* In this case we suppose that A contains no trivial right ideal of R, and show that $3A = (0)$. Our hypotheses imply that A is semiprime and purely alternative, by [9, 3.2d]. Hence $N(A) = Z(A)$ by [9, 3.2g]. Since A is both nil and semiprime, we have $Z(A) = (0)$, whence $N(A) = (0)$. So $3A = (0)$ by [9, 3.2e].

(Note that for this case we did not need the "bounded index" part of our hypothesis. Indeed, we have just shown that a semiprime purely alternative ring is free of nil ideals if it is free of 3-torsion.)

2. *R is not A-purely alternative.* In this case we show that A contains a trivial right ideal of R. Set $B = A \cap U(R)$. Thus B is a nonzero right

ideal of R, and $B \subseteq N(R)$. We can now repeat the associative arguments word-for-word (e.g. see [3], p. 385) to conclude that B, and hence also A, contains a trivial right ideal of R.

COROLLARY 2.10. *Suppose R is nil of bounded index, and free of 3-torsion. Then R is a Baer-radical ring.*

Proof. If B is the Baer radical of R, and $R' = R - B$, then R' is nil of bounded index and semiprime. So by 2.9 $3R' = (0)$, or in other words $3R \subseteq B$. We can now use our hypothesis on characteristic to conclude that $R = B$. A full proof requires detailed examination of $B(R)$, and is rather messy, although routine. We therefore omit the details.

3. RINGS WITH REASONABLE ELEMENTS

3.1. In order to apply Proposition 2.5 to rings (instead of algebras), we will need the relation $N(R) = Z(R)$. Actually, this condition occurs quite naturally in the theory of weakly prime rings. We recall from [9, Section 2], that R is *weakly prime* provided

(i) R is semiprime;

(ii) R is purely alternative;

(iii) $Z(R)$ contains no R-zero-divisors.

If R is weakly prime, then R also satisfies

(iv) $N(R) = Z(R)$.

Indeed, (iv) follows from (i) and (ii). For a more detailed statement of the relations between these conditions, we introduce a further condition (v), as follows:

(v) $(0) = T^2 \leqslant T \leqslant R$ and $T \cap Z(R) = (0)$ implies $T \cap N(R) = (0)$.

Clearly (v) is weaker than (iv). We now have

Note 3.2. *Suppose R is not associative. Then the following are equivalent:*

(a) *R satisfies* (ii) *and* (iii) *and* (iv);

(b) *R satisfies* (iii) *and* (iv);

(c) *R satisfies* (ii) *and* (iii) *and* (v);

(d) *$N(R)$ contains no R-zero-divisor.*

Sketch proof. (b) $\rightarrow$ (ii). If $u \in U(R)$, then $Du = (0)$. By (iv), $u \in Z(R)$, and since $D \neq (0)$ (iii) yields $u = 0$. So $U = (0)$.

(c) $\rightarrow$ (iv). In the presence of (iii), (v) reduces to: If $(0) = T^2 \leqslant T \leqslant R$, then $T \cap N(R) = (0)$. This together with (ii) implies (iv), by 6.8(iii) of [7].

(b) $\rightarrow$ (d). Immediate.

(d) $\rightarrow$ (iv). By 7.4 of [7], any ring R satisfies the identity $(a, b, c)(d, n)^2 = 0$ for $n \in N$. If $D(R) \neq (0)$, (d) then yields $(d, n) = 0$.

I do not know whether the above sets of conditions are equivalent to (ii) + (iii).

3.3. Since (v) is a weakening of (i) as well as of (iv), the equivalence of (b) and (c) above allows us to regard (iii) + (iv) as a fragment of the conditions for weak primeness.

We now recall that if R satisfies (iii) and $Z(R) \neq (0)$, then we can form the quotient field Z' of Z, and the algebra $R' = R \otimes_Z Z'$ over the field Z', and that R is in a certain sense 'tightly' imbedded in R' [9, 2.5].

PROPOSITION 3.4. *Suppose R satisfies* (iii) *and* (iv), *and has a reasonable element. Then $R' = R \otimes_Z Z'$ exists and is an algebra over Z' satisfying all the conclusions of Proposition* 2.5.

Proof. If $\langle a, b\rangle$ is a reasonable pair, we form $0 \neq (a, b)^4 \in N(R) = Z(R)$. Thus $Z(R) \neq (0)$, and we can form R'. By [9, 2.6] R' inherits (iii) and (iv), $\langle a, b\rangle$ is a reasonable pair in R', and $N(R') = Z(R') = Z'$, a field over which R' is an algebra. The conclusion now follows from Proposition 2.5.

EXAMPLE 3.5. Let A be the algebra over any field F constructed by Dorofeyev [2]. It is easily verified that $N(A) = (0)$, so that A trivially satisfies (ii), (iii), and (iv). But A is not quadratic over F since, for example, $a^3 = 0$ but $a^2 \neq 0$ for $a = x_1 + x_2 + x_3x_4$. Thus the hypothesis in Proposition 3.4 that R has a reasonable element cannot be dropped. Nor can it be replaced by the hypothesis $Z(R) \neq (0)$, in view of the example $A + F \cdot 1$.

The 'hope' mentioned in [9, 5.1α] was that (ii) and (iii) imply (iv) and the existence of a reasonable element. This example shows that hope for the latter was misplaced, while 3.4 justifies the other statements in [9, 5.1α].

As a corollary to Proposition 3.4 we have

THEOREM A. *Suppose R is weakly prime and has a reasonable element. Then R is a* CD *ring.*

Proof. In view of Note 3.2, R satisfies the conditions of Proposition 3.4. Since R is semiprime, R' is semiprime by [9, 2.6e]. So R' is a CD algebra by Corollary 2.8, and R is a CD ring.

3.6. We will now try to improve Theorem A by weakening the conditions of weak primeness, and localizing them to the associator ideal $D(R)$, just as we did in [9]. We will have continual need of the following:

LEMMA 3.6. *Suppose R satisfies*

(i_D) *R is D-semiprime.*

Then D is semiprime; D has no nonzero associative ideals; $Z(D) = D \cap Z(R) = D \cap N(R)$; any ideal A of R disjoint from D lies in $U(R)$.

Proof. This follows from [9, 4.5 and 3.2a].

THEOREM B. *Suppose R satisfies*

(i_D) *R is D-semiprime;*

(iii_D) *$D \cap Z(R)$ contains no R-zero-divisor;*

(r_D) *D contains a reasonable element of R.*

Then R is a CD *ring.*

Proof. By Lemma 3.6, D is weakly prime. If $d \in D$ is a reasonable element of R, say $u^4 \neq 0$ for $u = (d, r)$. Then $0 \neq \beta = u^4 \in D \cap N(R) = D \cap Z(R)$ by Lemma 3.6. Hence $(d, \beta r)^4 = \beta^4(d, r)^4 = \beta^5 \neq 0$ by (iii_D) and $\beta r \in D$ implies that d is a reasonable element *of the ring D*. So by Theorem A D is a CD ring. If $0 \neq \alpha \in D \cap Z(R)$, then $\alpha U = DU = (0)$ implies $U = (0)$; so by Lemma 3.6 D is essential in R. Since D is prime, it follows that R is prime, hence weakly prime, since $D \neq (0)$. Since R has a reasonable element, it follows from Theorem A that R is a CD ring.

COROLLARY 3.7. *Suppose R is a central semiprime alternative algebra over any field F. If $D(R)$ contains a reasonable element of R, then R is a* CD *algebra over F.*

Proof. The conditions imply that R satisfies (i) and (iii), and we can apply Theorem B and the argument of [9, 4.3].

THEOREM C. *Suppose R satisfies*

(i_D) *R is D-semiprime;*

(ii) *R is purely alternative;*

(iii''_D) *$D \cap Z(R)$ has no zero-divisors (as a subring);*

(r'_D) *If $(0) \neq A \leqslant R$ and $A \subseteq D$, then A contains a reasonable element of R.*

Then R is a CD *ring or* (0).

Proof. Given $A \leqslant R$, we have $(0) \neq A_0 = A \cap D \leqslant R$ by (ii) and Lemma 3.6. By (r'_D) we may choose $a \in A_0$, $r \in R$ such that $\alpha = (a, r)^4 \neq 0$. Then $\alpha \in D \cap N(R) \subseteq Z(R)$ by Lemma 3.6, and since $A \leqslant R$ we also have $\alpha \in A$. Now, given $A, B \leqslant R$, choose $0 \neq \alpha \in A \cap Z(R)$; $0 \neq \beta \in B \cap Z(R)$. Then by (iii''_D) $0 \neq \alpha\beta \in AB$. So R is prime, hence weakly prime. If $D = (0)$ then by (ii) $R = (0)$; otherwise (r'_D) yields that R has a reasonable element. But then R is a CD ring by Theorem A.

THEOREM D. *Suppose R satisfies*

(i_D) *R is D-semiprime;*

(iii'_D) *$D \cap Z(R)$ contains no $N(R)$-zero-divisor;*

(r'_D) *(as for Theorem* C*).*

Then R is a CD *ring or associative.*

Proof. If $D = (0)$ then R is associative. Otherwise, (r'_D) implies (r_D), so we can find $0 \neq \alpha \in D \cap N(R) = D \cap Z(R)$ by Lemma 3.6. Then (iii'_D) and $\alpha U = (0)$ yield (ii). The result now follows from Theorem C.

3.8. There is a strong formal analogy between the theorems of this section and the theorems of [9] bearing the same letters, as follows. Let us write (3_A) for "$3A \neq (0)$", and (r_A) for "A contains a reasonable element of R". Then in Theorems A we assume (i), (ii), (iii), and either (3_R) or (r_R). In Theorems B we assume (i_D), (iii_D), and either (3_D) or (r_D). In Theorems C we assume (i_D), (ii), (iii''_D), and either (in effect) (3_A) for each $(0) \neq A \leqslant R$ with $A \subseteq D$, or (r_A) for each such A. In Theorems D the 'side' condition is as in Theorems C.

The corollaries 3.7 in this paper and 4.3 in [9] can be stated as follows: Suppose R is a central semiprime alternative algebra over any field F. If R satisfies (r_D) or (3_D), then R is a CD algebra over F. Finally, the analog of Theorem E of [9] holds with (r'_D) for side condition, although we do not deal with that matter in this paper.

3.9. It remains to show that our conditions on reasonable elements are not excessive. We give examples to show that if there exists an exceptional ring S, as in [9, Theorem J], then our results break down if we weaken the restrictions.

In Theorem A we cannot drop (r_R) in view of the example $R = S$. (Note that the p constructed for Theorem J(d) was $p(x, y) = (x, y)^4$.)

In Theorem B we cannot relax (r_D) to (r_R) in view of the example $R = S \oplus V$. Here V is the algebra of finitely nonzero infinite matrices over a field F, and we can check that V has reasonable elements. This example satisfies (i) and (iii) globally.

In Theorems C and D (and E) we cannot relax (r'_D) to (r_D), in view of the example $R = S \oplus C$, where C is a CD algebra.

Finally, in 3.7 we cannot relax (r_D) to (r_R) in view of the following example. Let V be as above, but with $F = GF3$. Then construct R by adjoining a unity over F to the ideal direct sum $S \oplus V$.

4. Rings without Nil Ideals

4.1. The results of Section 3 all require conditions on elements. Now such conditions are usually less convenient than conditions on ideals. In this section, therefore, we give results in which the conditions concerning reasonable elements are in each case replaced by some condition on ideals stronger than the condition (i) of semiprimeness.

We start with two key lemmas, neither of which is really new.

Lemma 4.2. *Suppose R has no reasonable elements. Then*

(a) *The nil elements of R form an ideal K of R.*

(b) *If $K = (0)$ then R is associative and commutative.*

Proof. (a) is due jointly to a number of authors, and is proved, for example, in [3, pp. 386–387].

(b) If R has no reasonable elements, then for all $x, y \in R$ we have $(x, y)^4 = 0$, or $(x, y) \in K$. Thus if $K = (0)$ then R is commutative. To show that R is also associative, it is enough to show that every associator falls in K. So the following (known) result completes the proof.

Note 4.3. *Given $p \in R$, suppose $(p, R) = (0)$. Then $3p \in N(R)$ and $p^3 \in N(R)$. If R is commutative, then $(p, q, r)^3 = 0$ for all $p, q, r \in R$.*

Proof. That $3p \in N(R)$ follows from the identity

$$0 = (p, qr) - q(p, r) - (p, q)r = 3(p, q, r).$$

for all $q, r \in R$. Next, an easy application of Kleinfeld's function f shows that in any ring we have $(p^3, q, r) = p^2t + ptp + tp^2$, where $t = (p, q, r)$. So in this case we have $(p^3, q, r) = 3p^2t = p^2 \cdot 3t = 0$ by the above. Since this holds for all $q, r \in R$, we have $p^3 \in N(R)$.

Now for given $p, q, r \in R$ set $t = pq \cdot r - p \cdot qr = a - b$, say. Then $t^3 = (a - b)^3 = a^3 - b^3 - 3ab(a - b)$, where we use Artin's theorem and commutativity. But $3ab(a - b) = ab \cdot 3t = 0$. So $t^3 = a^3 - b^3$. Now again by Artin's theorem, $a^3 = (pq \cdot r)^3 = (pq)^3 r^3 = p^3q^3 \cdot r^3$, and similarly $b^3 = p^3 \cdot q^3r^3$. Thus $t^3 = (p^3, q^3, r^3) = 0$ since $p^3 \in N(R)$.

Lemma 4.2 shows how nil ideals enter naturally into this subject. Let us write $K(R)$ for the maximum nil ideal of any given ring R. Then we have

LEMMA 4.4. *If $A \leqslant R$, then $K(A) = A \cap K(R)$.*

Proof. The class $\mathscr{K}$ of nil (alternative) rings is a radical class, in the sense that $\mathscr{K}$ is homomorphically closed, and each ring R has an ideal $K(R) \in \mathscr{K}$ such that $R - K(R)$ has no nonzero ideal in $\mathscr{K}$. Now in the universe $\mathscr{A}$ of all (alternative) rings, if $\mathscr{M}$ is any radical class, and $A \leqslant R \in \mathscr{A}$, it is known that $M(R) = (0)$ implies $M(A) = (0)$. It follows easily that if $\mathscr{M}$ is hereditary, in the sense that $A \leqslant R \in \mathscr{M}$ implies $A \in \mathscr{M}$, then for all $I \leqslant S \in \mathscr{A}$ we have $M(I) = I \cap M(S)$. For all this see [1, Corollary 1 to Theorem 2, and Lemmas 1 and 2].

(We write $\mathscr{K}$ and K for the nil radical class and the nil radical (operator), because it was first investigated, for associative rings, by Koethe, 1930. The letter N is already preempted for the nucleus.)

We now produce analogs of the results of Section 3, where the conditions on reasonable elements are replaced in each case by some condition on the nonexistence of nil ideals.

THEOREM A′. *Suppose R is weakly prime and $K(R) = (0)$. Then R is a* CD *ring or* (0).

Proof. If R has a reasonable element we are done by Theorem A. Otherwise 4.2 shows that $R = U(R)$, and (ii) then yields $R = (0)$.

4.5. Theorem A′ allows us to dispose of a question concerning the definition of CD rings, as follows. We recall that R is a CD ring provided R is a (nonassociative) ring with $Z(R) \neq (0)$, satisfying (iii), and such that $R' = R \otimes_Z Z'$ is a CD algebra over Z'. An equally natural way of defining the notion 'CD ring' is given in the following

DEFINITION 4.5. We say the (nonassociative) ring R is a *CD_0 ring* provided there exists a CD algebra S over some field F, and an imbedding θ of R into S such that $S = F \cdot R\theta$.

Fortunately the two definitions are equivalent, in view of

PROPOSITION 4.5. *R is a* CD_0 *ring if and only if R is a* CD_0 *ring.*

Proof. (a) Suppose R is a CD ring with center Z, and set $S = R \otimes_Z Z'$. Then S is a CD algebra over the field $F = Z'$, and there is an imbedding

$\theta : r \to r \otimes 1$ of R into S. By [9, 2.6b] each $s \in S$ can be written in the form $s = r \otimes \alpha^{-1} = \alpha^{-1} \cdot r\theta$ for suitable $r \in R$; $0 \neq \alpha \in Z \subseteq Z'$. Thus $S = F \cdot R\theta$.

(b) Suppose R is a CD_0 ring, and θ imbeds R in the CD algebra S over F, with $S = F \cdot R\theta$. Let us identify R with $R\theta$. Clearly R is alternative. Given $s \in S$, we can write $s = \sum \alpha_i a_i$ with $\alpha_i \in F$; $a_i \in R$. Hence we easily see that $N(R) = R \cap N(S) = R \cap Z(S) = Z(R)$, so that R satisfies (iii). Next, if $A \leqslant R$, we easily see that $FA \leqslant S$. If $A^2 = (0)$ or $A \subseteq N(R)$, then $(FA)^2 = (0)$ or $FA \subseteq N(S)$, so that R satisfies (i) and (ii), and so is weakly prime. If A is a nil ideal of R, then $t(a) = 0$ for each $a \in A$, where t is the trace function on S defined in Proposition 2.5. Thus $FA \leqslant S$, and $t(x) = 0$ for each $x \in FA$. Since $t \neq 0$ on S and S is simple, we conclude that $A \subseteq FA = (0)$. So R is free of nil ideals, and by Theorem A′ R is a CD ring.

It would be interesting to have an elementary proof of (b).

We omit the analog of Theorem B, since it is subsumed under Theorem D′ below.

THEOREM C′. *Suppose R satisfies*

(1_D) $D \cap K(R) = (0)$;

(ii) *R is purely alternative*;

(iii''_D) *$D \cap Z(R)$ contains no zero-divisors (as a subring).*

Then R is a CD *ring or* (0).

Proof. By Lemma 4.4 $K(D) = (0)$. If $A \leqslant D$ has no reasonable element, then Lemma 4.2 together with $K(A) = A \cap K(D) = (0)$ shows that A is associative, hence (0) by Lemma 3.6. Thus, every nonzero ideal of D has a reasonable element, and the theorem then follows from Theorem C.

THEOREM D′. *Suppose R satisfies*

(1_D) $D \cap K(R) = (0)$;

(iii'_D) *$D \cap Z(R)$ contains no $N(R)$-zero-divisor.*

Then R is a CD *ring or associative.*

Proof. Exactly as in the proof of Theorem C′, we show that the condition (r'_D) of Theorem D holds.

COROLLARY 4.6. *If R is a central alternative algebra over any field F, and $K(R) = (0)$, then R is associative or a* CD *algebra over F.*

Proof. This result follows from Theorem D′ in exactly the same way as Corollary 3.7 follows from Theorem B.

4.7. We now relax the conditions on nil ideals to conditions on locally nilpotent ideals. Our improvements need a lemma which, roughly speaking, requires char $R \neq 2$. In order to avoid restrictions on characteristic in the theorems below, we are therefore forced to appeal to the results of [9], which solve all our problems except in char 3, where note $3 \neq 2$. This inelegant procedure seems unavoidable in the present state of our knowledge, and for aesthetic reasons the reader may prefer to rest content with the (weaker) results given above.

LEMMA 4.8. *If R has no reasonable elements and is free of 2-torsion ($2x = 0$ implies $x = 0$), then $K(R)$ is locally nilpotent.*

Proof. R satisfies the p.i. $(x, y)^4 = 0$, which is not a consequence of associativity, and has leading coefficient 1. The result then follows from results essentially due to Shirshov (see [8, p. 712], including the footnote).

THEOREM A″. *Suppose R is weakly prime and free of locally nilpotent ideals. Then R is a* CD *ring or* (0).

Proof. By Theorem A we may suppose that R has no reasonable element, and by Theorem A of [9] that $3R = (0)$. Then by Lemma 4.8, $K(R)$ is locally nilpotent; so our hypothesis yields $K(R) = (0)$. But now Lemma 4.2 yields $R = U(R)$, and (ii) then yields $R = (0)$.

We omit the analog of Theorem B, since it is subsumed under Theorem D″ below.

THEOREM C″. *Suppose R satisfies*

(I_D) *no nonzero locally nilpotent ideal of R lies in D;*

(ii) *R is purely alternative;*

(iii''_D) *$D \cap Z(R)$ has no zero-divisors (as a subring).*

Then R is a CD *ring or* (0).

Proof. Let A be any ideal of R contained in D. We show that $A = (0)$ or A has a reasonable element.

Suppose first A is free of 2-torsion. If A has no reasonable element then by Lemma 4.8 $K(A)$ is locally nilpotent, and by Lemma 4.4 $K(A) = A \cap K(R)$ is an ideal of R. So by (I_D) $K(A) = (0)$. Then A is associative by Lemma 4.2, and $A = (0)$ by Lemma 3.6.

Suppose next A has 2-torsion. Then $(0) \neq A_2 = A \cap R_2 \leqslant R$, where we write B_2 for $\{b \in B\colon 2b = 0\}$. By 3.6 and [9, 3.2d] A_2 satisfies (i) and (ii). Then by [9, 3.2b], $Z(A_2) = A_2 \cap Z(R)$; so A_2 also satisfies (iii″). Finally, A_2 is obviously free of 3-torsion. So A_2 is a CD ring by Theorem C of [9], and so has a reasonable element. Thus A has a reasonable element also.

We now have all the hypotheses of Theorem C, and the result follows.

THEOREM D″. *Suppose R satisfies*

(I_D) *No nonzero locally nilpotent ideal of R lies in D;*

(iii'_D) *$D \cap Z(R)$ contains no $N(R)$-zero-divisor.*

Then R is a CD *ring or associative.*

Proof. Exactly as in the proof of Theorem C″ we show that the condition (r'_D) of Theorem D holds.

COROLLARY 4.8. *Suppose R is a central alternative algebra over any field F, and R is free of locally nilpotent ideals. Then R is associative or a* CD *algebra over F.*

4.9. The results of this paper throw some light on Theorem J of [9]. Suppose for our present purpose that some exceptional weakly prime ring exists. Then a ring exists having the properties (a)–(j) listed there. Examples were given to show that not every exceptional weakly prime ring satisfies (c), (e), or (f) [9, 8.2 and 8.4]. We can now see that any exceptional ring R satisfies most of the remaining conditions: (a); (b) by Theorem A of [9]; (d) by Theorem A of this paper; (g) because a minimal right or left ideal is a CD algebra, and these contain reasonable elements. It is not clear to me whether R necessarily satisfies (h).

Although R need not itself satisfy (e), (c), or, perhaps, (h), it will have an ideal I which satisfies these as well as the conditions (a), (b), (d), (g). For by Theorem A″ R has an ideal I satisfying (a) and (e). Then I inherits (b) and (d) from R, and as above I satisfies (g). Also $N(I) = Z(I) \subseteq Z(R)$ by semiprimeness of R, whence $N(I) = (0)$ since I is nil. Thus I satisfies (c). Since I is semiprime and satisfies (c), it follows that I also satisfies (h) [9, p. 249]. It is not clear to me whether any exceptional weakly prime ring necessarily has an ideal satisfying (f).

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